AN EXAMINATION OF THE EFFECT OF ATTACK VELOCITY ON THE OUTCOME OF LANCHESTER-TYPE ENGAGEMENTS WITH RANGE DEPENDENT KILL-RATES

By

James Francis Lloyd

United States Naval Postgraduate School

THESIS

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James Francis Lloyd, Jr.

Thesis Advisor: J. G. Taylor

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James Francis Lloyd, Jr.
Captain, United States Marine Corps
B.S., United States Naval Academy, 1963

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ABSTRACT

This thesis examines the effect of attack velocity on the outcome of Lanchester-type engagements between forces with range dependent kill-rates. Range dependent (linear and quadratic) kill-rates are considered, and analytic solutions to Lanchester-type equations are utilized in this study.

By varying the attack velocity, the effects on terminal force strengths are investigated for the case when an attacking force has the initial fighting strength superiority, and for the case when a defending force has the initial fighting strength superiority.

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I. INTRODUCTION

S. Bonder and colleagues [8] have described the necessity for military planners to evaluate, both comparatively and individually, the effectiveness and costs of proposed weapon systems and force structures. They have pointed out that Monte Carlo simulations or wargames have been used extensively in situations where adequate descriptive theory has not existed. The shortcomings of the Monte Carlo simulations and wargames in terms of cost and degree of difficulty of analysis are noted, as are the advantages and disadvantages of analytic models. One of the disadvantages of analytic models is the limited number of them available. The best known of the analytic models which are useful in weapon system and force structure evaluation are the Lanchester theories of combat.

Lanchester equations are sets of simultaneous differential equations which describe deterministically the attrition of two opposing homogeneous forces [2]. The more well known of the Lanchester equations are those which describe combat between two forces employing aimed fire against each other; they are:

$$\frac{dx}{dt} = -\alpha y(t) \tag{1}$$

$$\frac{dy}{dt} = -\beta x(t) \tag{2}$$

where:

α = constant rate at which a single y unit destroys a single x unit.

β = constant rate at which a single x unit destroys a single y unit.

$x(t)$ = number of X force survivors at time t.

$y(t)$ = number of Y force survivors at time t.

The quantities α and β are known as the attrition-rate coefficients, or kill-rates. The state solution of (1) and (2), i.e., the solution with the independent variable, time, removed, is:

$$\beta(X^2 - x^2) = \alpha (Y^2 - y^2), \tag{3}$$

where X and Y are the initial force strengths. This is referred to as the Lanchester square law. If the forces are evenly matched during an engagement, x and y, the number of survivors, approach zero together. Equation (3) then becomes:

$$\beta X^2 = \alpha Y^2 , \tag{4}$$

which is the parity condition between forces. It is the condition for a draw.

The above Lanchester-type equations, (1) and (2), are based upon the following assumptions:

1. Opposing forces consist of homogeneous units.
2. All units of each force are within range of all weapons of all units of the opposing force.
3. Each force has perfect intelligence; i.e., the exact location of each unit of the opposing force

is known; and when an opposing unit is destroyed, fire is immediately shifted to a surviving unit.

4. Fire is distributed uniformly over all surviving targets.

In light of these necessary assumptions, several deficiencies in the model exist, prohibiting the use of Lanchester equations as effective planning tools. Only problems concerning forces composed of homogeneous units can be solved generally. It is sometimes difficult to theoretically predict the attrition-rate coefficients of proposed weapon systems. The effects of mobility on the weapon system and on the attrition-rate coefficient have not been generally incorporated into the Lanchester model [8].

Weiss [9] presented an extension of Lanchester theory of combat which included the relative movement of forces, and thereby allowed time and space to be "traded" for casualties. This extension was an attempt to incorporate the fact that the attrition-rate coefficients are dependent on force separation. One of Weiss's assumptions, however, was that during an engagement, force separation reached an equilibrium point, and that a force would advance or retreat based on a comparison of the actual casualty rate with some predetermined tolerable casualty rate. Thus, if a force were receiving casualties at a rate higher than the predetermined acceptable rate, that force would retreat; likewise, the other force would advance to maintain its equilibrium casualty rate. Weiss further assumed that the

time required for the forces to respond to fluctuations in their casualty rates was small compared to the time required for the forces to close. Therefore, there was no change in force separation, and hence, no change in the attrition-rate coefficients. Movement of forces was included in the model, but the effect of force separation on the attrition-rate coefficients was disregarded.

Bonder [1], [3], [8], has further developed Weiss's extensions of Lanchester equations to investigate the effects of mobility and range dependencies of weapon systems. He has done this by formulating a model which considers mobility and the influence of range on the attrition-rate coefficients. The attrition equations of this model are given by:

$$\frac{dx}{dt} = -\alpha(r)y(t) \tag{5}$$

$$\frac{dy}{dt} = -\beta(r)x(t) \ , \tag{6}$$

where the terms are defined as for equations (1) and (2), with the exception that the attrition-rate coefficients are no longer constant, but now vary with the range between forces.

Three forms of the attrition-rate coefficient were considered by Bonder. They were:

$$\alpha(r)=k_\alpha(R_e-r) \ , \quad \beta(r)=k_\beta(R_e-r) \qquad \text{(linear form)} \ (7)$$

$$\alpha(r)=\alpha_0(1-r/R_e)^2 \ , \quad \beta(r)=\beta_0(1-r/R_e)^2 \quad \text{(quadratic form)} \ (8)$$

$$\alpha(r)=\alpha_0/2[1+\cos(\pi r/R_e)], \quad \beta(r)=\beta_0/2[1+\cos(\pi r/R_e)] \quad \text{(cosine form)} \ (9)$$

where:

r = force separation,

R_e = effective range of both forces,

k_α, k_β = constants,

α_0, β_0 = attrition-rate coefficients at $r = 0$.

Mobility was incorporated into the model by transforming equations (5) and (6) into second order differential equations with r as the independent variable [1]. By assuming that the effective range, R_e, was the same for both forces, and the attrition-rate coefficients were of the same form, Bonder was able to obtain an analytic solution for the average force strengths, x(r) and y(r), for decreasing values of force separation [1], [3], [8].

Although Bonder's model is much more appropriate than the classical Lanchester theory for the analysis of mobile weapon systems, its use is delimited by the assumptions necessary to obtain the solution for x(r) and y(r). Taylor [6], using Bonder's model, employed a different mathematical approach in obtaining a solution to (5) and (6). The solution he obtained is much more general than that obtained by Bonder; the assumptions necessary for Bonder's solution are not required for Taylor's solution. Taylor's solution is in the form of an infinite series, with time as the independent variable. Taylor has previously shown [7] that it is inconsequential if either force separation or time is considered as the independent variable, one form being derived from the other. Thus, a general solution to Lanchester equations is now available for those situations in which the

opposing forces have different effective ranges and/or different range dependencies of attrition-rate coefficients.

Of particular interest is Taylor's solution to the Lanchester equations when the forces have attrition-rate coefficients of the following form:

$$a(t)=k_a t^m \tag{10}$$

$$b(t)=k_b t^n. \tag{11}$$

The solution to the attrition equations

$$\frac{dx}{dt} = -a(t)y(t)$$

$$\frac{dy}{dt} = -b(t)x(t)$$

with a(t) and b(t) given by (10) and (11) respectively is:

$$x(t) = t^{\frac{1+m}{2}} \left\{ X\Gamma(1-p) \left[\frac{\sqrt{k_a k_b}}{2s}\right]^{p} I_{-p}\left[\frac{\sqrt{k_a k_b}}{s} t^s\right] - \frac{k_a Y\ \Gamma(1+p)}{(m+1)} \left[\frac{\sqrt{k_a k_b}}{2s}\right]^{-p} I_p\left[\frac{\sqrt{k_a k_b}}{s} t^s\right]\right\} \tag{12}$$

$$y(t) = t^{\frac{1+n}{2}} \left\{ Y\Gamma(1-q) \left[\frac{\sqrt{k_a k_b}}{2s}\right]^{q} I_{-q}\left[\frac{\sqrt{k_a k_b}}{s} t^s\right] - \frac{k_b X\ \Gamma(1+q)}{(n+1)} \left[\frac{\sqrt{k_a k_b}}{2s}\right]^{-q} I_q\left[\frac{\sqrt{k_a k_b}}{s} t^s\right]\right\}, \tag{13}$$

where

$$s = \frac{(m+n+2)}{2}, \quad p = \frac{m+1}{(m+n+2)}, \quad p+q = 1, \qquad (14)$$

and $I_\nu(x)$ is the modified Bessel function of the first kind and order ν [6]. The solution is now in terms of tabulated functions.

Bonder's model and Taylor's solution for average force strengths given by (12) and (13) were used to extend Bonder's study [3], [8], of the effect of mobility on the outcome of a Lanchester type engagement when both forces had the same effective range. The attrition-rate coefficients were dependent on the force separation, but not necessarily in the same manner.

II. METHOD OF STUDY

In investigating the effects of mobility on the outcome of a Lanchester-type engagement, the following questions were considered:

1. How were the terminal force strengths affected by the attack velocity?
2. How did the form of the attrition-rate coefficient contribute to the effects of attack velocity?

A. SCENARIO

The following scenario was used in an attempt to answer the above questions. Two homogeneous forces, an X force and a Y force, were engaged in combat. The X force was the attacking force, while the Y force was defending a fixed position. Both forces had attrition-rate coefficients which, in most cases, varied with force separation. This force separation closed at a constant velocity. The range at which the battle commenced was the effective range for both forces. The attrition-rate coefficients used were the average values for each value of force separation, i.e., stochastic variations in attrition rates at a specific range were disregarded [8]. The attrition rates used were of the following form:

$$\alpha(r) = \alpha_0(1-r/R_e)^m \tag{15}$$

$$\beta(r) = \beta_0(1-r/R_e)^n. \tag{16}$$

Equations (15) and (16) can be shown to be equivalent to (10) and (11) be letting

$$k_a = \alpha_0 \left(\frac{V}{R_o} \right)^m, \quad k_b = \beta_0 \left(\frac{V}{R_o} \right)^n,$$

and $$t = \frac{R_o - r}{V},$$

where R_o is the range at which the battle begins (R_o=R_e), and V is the attack velocity. The force attrition for this scenario is given by (5) and (6). The necessary assumptions inherent in the Lanchester theory of combat given above are applicable in this situation.

B. DESCRIPTION OF MODEL

The model used was based on Taylor's solution, given in (12) and (13). Using the relationships that

$$r(t) = R_o - Vt, \quad k_a = \alpha_0 \left(\frac{V}{R_o} \right)^m, \quad k_b = \beta_0 \left(\frac{V}{R_o} \right)^n,$$

and $$\sqrt{k_a k_b} = \left(\frac{V}{R_o} \right)^{\frac{m+n}{2}} \sqrt{\alpha_0 \beta_0},$$

a solution with range as the independent variable was obtained. This solution is as follows:

$$x(r) = \left(1 - \frac{r}{R_o}\right)^{\frac{m+1}{2}} \left\{ X\Gamma(1-p) \left[\frac{R_o}{V} \frac{\sqrt{\alpha_0 \beta_0}}{2s} \right]^p \cdot \right. \tag{17}$$

$$I_{-p} \left[\frac{R_o}{V} \frac{\sqrt{\alpha_0 \beta_0}}{s} \left(1 - \frac{r}{R_o}\right)^s \right] - \frac{\alpha_0 Y \, \Gamma(1+p)}{(m+1)} \left(\frac{R_o}{V} \right) \cdot$$

$$\left[\frac{R_o}{V}\frac{\sqrt{\alpha_0\beta_0}}{2s}\right]^{-p} I_p\left[\frac{R_o}{V}\frac{\sqrt{\alpha_0\beta_0}}{s}\left(1-\frac{r}{R_o}\right)^s\right]\Bigg\}$$

and

$$y(r) = \left(1-\frac{r}{R_o}\right)^{\frac{n+1}{2}} \Bigg\{ Y\Gamma(1-q)\left[\frac{R_o}{V}\frac{\sqrt{\alpha_0\beta_0}}{2s}\right]^q \cdot \quad (18)$$

$$I_{-q}\left[\frac{R_o}{V}\frac{\sqrt{\alpha_0\beta_0}}{s}\left(1-\frac{r}{R_o}\right)^s\right] - \frac{\beta_0 X\ \Gamma(1+q)}{(n+1)}\left(\frac{R_o}{V}\right) \cdot$$

$$\left[\frac{R_o}{V}\frac{\sqrt{\alpha_0\beta_0}}{2s}\right]^{-q} I_q\left[\frac{R_o}{V}\frac{\sqrt{\alpha_0\beta_0}}{s}\left(1-\frac{r}{R_o}\right)^s\right]\Bigg\} ,$$

where p, s, and q are as previously defined in (14). Again, this solution is in terms of tabulated functions. There are, however, few tables of modified Bessel functions of the first kind of fractional order. These tables are available in reference [5]. The modified Bessel function of the first kind of order p has the power series expansion [4]

$$I_p(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k+p}}{k!\,(k+p)!} \quad .$$

Using this form of the Bessel function, equations (17) and (18) can be rewritten as

$$x(r) = X\Gamma(1-p)\left\{\sum_{k=0}^{\infty}\left(\frac{R_o}{V}\frac{\sqrt{\alpha_0\beta_0}}{2s}\right)^{2k}\frac{\left(1-\frac{r}{R_o}\right)^{2ks}}{k!\,\Gamma(k+1-p)}\right\} \quad (19)$$

$$- \frac{\alpha_0 Y \, \Gamma(1+p)}{m+1} \left(\frac{R_o}{V}\right) \left\{ \sum_{k=0}^{\infty} \left(\frac{R_o}{V} \frac{\sqrt{\alpha_0 \beta_0}}{2s}\right)^{2k} \cdot \frac{\left(1 - \frac{r}{R_o}\right)^{2(ks+ps)}}{k!\,\Gamma(k+1+p)} \right\}$$

and

$$y(r) = Y\Gamma(1-q) \left\{ \sum_{k=0}^{\infty} \left(\frac{R_o}{V} \frac{\sqrt{\alpha_0 \beta_0}}{2s}\right)^{2k} \frac{\left(1 - \frac{r}{R_o}\right)^{2ks}}{k!\,(k+1-q)} \right\} \quad (20)$$

$$- \frac{\beta_0 X \, \Gamma(1+q)}{n+1} \left(\frac{R_o}{V}\right) \left\{ \sum_{k=0}^{\infty} \left(\frac{R_o}{V} \frac{\sqrt{\alpha_0 \beta_0}}{2s}\right)^{2k} \cdot \frac{\left(1 - \frac{r}{R_o}\right)^{2(ks+qs)}}{k!\,\Gamma(k+1+q)} \right\}.$$

A computer program which evaluated equations (19) and (20) was used to obtain numerical results.

C. INPUT PARAMETERS

Two sets of numerical results were computed using this model. In one case, Case I, the parameters were selected such that the X force had a larger initial strength than the Y force, but the Y force had greater fighting strength in the classical Lanchester sense, (i.e., $\alpha_0 Y^2 > \beta_0 X^2$). In the other case, Case II, the X force had both an initial strength advantage and a fighting strength advantage ($\beta_0 X^2 > \alpha_0 Y^2$). The parameter values used in each of these cases are shown in Table I. In obtaining numerical results in each case,

some of the parameters were held constant, while other parameters were varied. The parameters which were held constant were the initial force strengths, the values of α_0 and β_0, and the effective range; the parameters which were varied were the attack velocity, and the exponents in the attrition-rate coefficients, m and n. For each combination of m and n, (19) and (20) were evaluated for values of attack velocity varying from 0.5 meters per second to 25 meters per second.

The values of α_0 and β_0 were selected arbitrarily; none of the results of studies of the attrition-rate coefficient [1], [8], were applied in obtaining these values. The values selected are, however, the same order of magnitude as those used by Bonder in his work. Figure 1 shows how the attrition-rate coefficients vary with range for different values of m and n.

D. USE OF THE MODEL

1. Computer Program

Numerical results for the model were obtained from a computer program written in FORTRAN IV language; this program is shown on pages 40 and 41. Equations (19) and (20) are particularly amenable to computer based evaluation. The infinite series terms in both equations were evaluated using recursive relationships. In describing the programming technique used in computing these infinite series terms, only equation (19) will be discussed. An analagous technique was used for the evaluation of the terms in (20).

Equation (19) can be rewritten as follows:

$$x(r) = X\Gamma(1-p)S_1 - \frac{\alpha_0 \gamma_1 \Gamma(1+p)}{(m+1)} \left(\frac{R_o}{V}\right) S^2 \quad ,$$

where

$$S_1 = \sum_{k=0}^{\infty} T_{1k} \quad \text{and} \quad S_2 = \sum_{k=0}^{\infty} T_{2k} \; .$$

Let

$$C = \left(\frac{R_o}{V} \frac{\sqrt{\alpha_0 \beta_0}}{2s} \right)^2 \left(1 - \frac{r}{R_o} \right)^{m+n+2} .$$

Now

$$T_{10} = \frac{1}{\Gamma(1-p)}$$

and

$$T_{1k} = C \times \frac{\Gamma(k-p)}{k \times \Gamma(k+1-p)} \times T_{1,k-1} \quad , \quad \text{for } k \geq 1 \; .$$

Let

$$D = \left(\frac{R_o}{V} \; \frac{\sqrt{\alpha_0 \beta_0}}{2s} \right)^2 \left(1 - \frac{r}{R_o} \right)^{2ps} .$$

Now

$$T_{20} = \frac{\left(1 - \frac{r}{R_o} \right)^{2ps}}{\Gamma(1+p)} \quad , \quad \text{and}$$

$$T_{2k} = D \times \frac{\Gamma(k+p)}{k \times \Gamma(k+1+p)} \times \frac{\left(1 - \frac{r}{R_o} \right)^{2ks}}{\left(1 - \frac{r}{R_o} \right)^{2(k-1)s+2ps}} \times T_{2,k-1}$$

for $k \geq 1$.

Thus the terms for S_1 and S_2 were obtained from the recursive relationships for T_{1k} and T_{2k}. In the program, S_1 and S_2 were computed as follows:

$$S_i = \sum_{k=1}^{N} T_{ik}$$

where N is the smallest integer such that $T_{iN} \leq 1\text{x}10^{-6}$.

The average force strength for both forces was computed for decreasing incremental values of force separation. The program terminated when either the X force or the Y force was completely annihilated, or when the X force overran the Y force position, (i.e., r=0).

2. Computer Program Output

The computer program gave the average X and Y force strengths for every 20 meters of decreasing force separation. From this information, the values of force strengths and force separations at the end of the engagement were obtained. The values of terminal force strength and final positions of forces were obtained for each combination of values of m and n, and for each value of attack velocity considered.

III. DISCUSSION OF OUTPUT AND RESULTS

Figures 2 through 10 show the effects of attack velocity on the terminal force strengths for each combination of parameters considered. The graphs readily indicate that the effect of attack speed on the outcome of the engagement is dependent on the parameter values used, particularly the values of m and n, which determine the form of the attrition-rate coefficients.

The classical Lanchester results are shown for Cases I and II in Figure 2. When m=n=0, the attrition-rate coefficients are constant throughout the engagement. Note that in Case I, i.e., when the Y force has greater fighting strength, the X force was annihilated before it reached Y's position when it attacked at speeds of less than 20 meters per second. At higher attack speeds, the X force was able to reach Y's position, but Y had numerical, and hence, fighting superiority. In Case II, with m=n=0, the X force dominated the engagement throughout. At attack speeds of less than 16 meters per second, the Y force was completely destroyed before the X force reached its position. At speeds greater than 16 meters per second, there were Y force survivors when the X force reached the defended position. As the attack speed increased, the number of X and Y force survivors increased, but the X force maintained numerical and fighting superiority. Thus, an increase in attack speed caused an increase in terminal strength in both Case I and Case II. The maximum velocity for which one side

was annihilated corresponds to the time required for annihilation, as obtained from the time solution of equations (1) and (2) [2]. Thus, increasing the attack velocity merely shortens the length of the engagement. Since the Lanchester square law holds in this case, a decrease in the duration of the battle results in an increase in the number of survivors of both forces. Thus, by attacking at a very high attack speed, a force with initial numerical superiority can obtain terminal numerical superiority.

When m=n=1, both attrition-rate coefficients vary linearly with force separation. Figure 3 illustrates how the outcome of the engagement in Case I and Case II was affected by the attack velocity. In Case I, the X force was annihilated before it was able to reach Y's position when it attacked at speeds less than 10 meters per second. At velocities greater than 10 meters per second, the number of X force survivors increased much faster than the number of Y force survivors. Finally, at velocities greater than approximately 24 meters per second, the number of X survivors exceeded the number of Y survivors, resulting in X force numerical superiority. In Case II, an increase in attack velocity resulted in an increase in the number of survivors for both forces, and a prolongation of the engagement. The X force maintained its terminal numerical and fighting superiority for all values of attack velocity. The situation when m=n=2, as shown in Figure 4, is similar to the case when m=n=1. In Case I, the X force achieved terminal numerical

superiority by attacking at a velocity greater than approximately 16 meters per second. In Case II, an increase in attack velocity caused an increase in the number of survivors for both forces, but the X force maintained its terminal strength advantage at all values of attack speed.

Thus, when both attrition-rate coefficients vary in a linear or quadratic manner, an attacking force can compensate for its lack of fighting strength superiority by attacking at a higher velocity. These results have been obtained by Bonder [3]. When the attacking force has the advantage in terms of fighting strength, it can increase its number of survivors by increasing its attack speed. This has the possibly adverse effect of causing an increase in the number of enemy survivors at the objective; the attacking force, however, is still the superior force, regardless of the attack speed. It would thus appear that the attacking force commander would have a choice of how to attack the defended position. A relatively slow attack would destroy the enemy prior to the attacker's arrival at the position, but would result in a larger number of casualties in the attacking force. Alternatively, a faster rate of attack would result in fewer friendly casualties, and more enemy survivors at the objective.

The increase in the number of suriviors of both forces is a direct consequence of the Lanchester square law; the square law holds when both linear and quadratic kill rates are considered. A result of the square law is that the two forces will always have the same strength values at a specific

time during a battle, regardless of the duration of the battle. Since an increase in attack velocity is equivalent to a decrease in the length of the battle, the number of survivors of both forces will increase. Thus, it is possible for a force with initial numerical superiority, but with firepower inferiority, to reach a defended position with numerical superiority by attacking at a higher speed. The reduction in the duration of the engagement prevents the defending force, with its superior firepower, from reducing the attacking force strength to an inferior level. A reduction in the duration of the battle is equivalent to a transformation of the time scale of the attrition process.

Figures 5 through 10 illustrate the outcome of the engagement when the opposing forces do not have attrition-rate coefficients of the same form. Figures 5 through 8 represent the cases when one of the forces has a constant attrition-rate coefficient (m or n = 0), and the other force has a linear or quadratic coefficient. As is readily obvious, when m=0, the attack velocity had no influence on the outcome of the engagement; the X force was annihilated in every instance except in Case II with m=0 and n=1. Then, as can be seen in Figure 5, there was a slight increase in the X force terminal strength at a velocity of about 25 meters per second. This increase was insignificant when compared with the Y force terminal strength.

When n=0, i.e., the X force attrition-rate coefficient is constant, the attack speed did influence the outcome of the engagement. Figures 7 and 8 depict these cases. The X

force had terminal numerical superiority in each of the engagements, and in each case, slower attack speeds resulted in more X force survivors. This is reasonable, since, as can be seen in Figure 1, the X force attrition-rate coefficient dominates the Y force coefficient at greater ranges between forces. Thus, at slower attack speeds, more time is spent at these greater ranges where the X force has the larger kill-rate. From this result, it is seen that it is advantageous to keep the enemy force in the best "kill zone."

A more practical, and common situation is when one force has a linear attrition-rate coefficient and the other force has a quadratic coefficient. Figures 9 and 10 illustrate these situations. In both Case I and Case II, with m=1 and n=2, the Y force had a large terminal numerical advantage at the slower attack speeds. However, the X force was able to reduce this superiority by attacking at a higher velocity. In Case I, the X force reduced its casualties by increasing its speed, even though it was never able to achieve a numerical advantage. However, in Case II, the increased attack velocity did result in X force numerical superiority. In both of these cases, the X force attrition-rate coefficient is very small at ranges near the effective range of both forces. For example, in Case I, with a force separation of 1900 meters, the X force coefficient, or kill-rate, is given by

$$\begin{aligned}\beta(1900) &= \beta_0 \, (1-\, r/R_0)^2 \\ &= .005 \, (1-\, 1900/2000)^2 \\ &= .0000125 \, .\end{aligned}$$

The corresponding Y force kill-rate at this range is

$$\alpha(1900) = .001 .$$

Thus, it is advantageous for the X force to close the range between forces as quickly as possible in order to reduce the time spent at ranges at which the Y force has such a distinct kill-rate advantage.

Figure 10 shows that when m=2 and n=1, the X force was the superior force in both Case I and Case II, regardless of the attack velocity. However, as in the case with m=1 or 2 and n=0, a reduced attack speed resulted in fewer X force casualties, while, at the same time, the Y force was annihilated. This is explained, as before, by the fact that during the early stages of the battle, the X force attrition-rate coefficient, together with the initial X force strength advantage, completely dominates the Y force coefficient and strength. Thus, it is more advantageous to attack at slower speeds.

IV. CONCLUSIONS

From the previous discussion, it is clear that the effects of attack velocity on the outcome of Lanchester-type engagements depend on the parameter values of the particular engagement, and especially on the form of the attrition-rate coefficients of each force. When the attrition-rate coefficient of the attacking force dominates that of the defender in the early stages of the battle, the attack should be conducted at a slow speed to take advantage of the kill-rate superiority enjoyed by the attacking force. Conversely, when the defender dominates the action in the early stages, it is advantageous to attack at a faster speed. Thus, when the kill-rate of a force does not dominate the kill-rate of the opposing force at all ranges, it is most advantageous for that force to hold the opposing force at ranges where it is subjected to the maximum kill-rate advantage. This implies the obvious result that the best tactic to employ is to keep the enemy in the "maximum killing zone."

The cases in which the attack speed had the least effect, if any, on the outcome of the battle, were the ones in which one, or both, of the forces had a constant attrition-rate coefficient. It is intuitively appealing that there would be few practical cases in which either of the forces involved would have a constant kill-rate. Therefore, in the more practical cases discussed, it appears that the speed of

attack does influence the outcome of a battle. However, no axiomatic statement describing this effect can be given, since the effect depends on the characteristics and capabilities of the opposing forces.

In an actual combat situation, the opposing forces might well have different effective ranges. An area for further investigation would be a study of the effects of attack velocity when forces do not have the same effective range. As previously stated, Taylor [6] has developed a solution to Bonder's model for the case when forces have different effective ranges and linear dependence of the attrition-rate coefficients. Taylor states [6] that his solution can be extended to situations where other range dependencies of the coefficients exist. Such an investigation using Taylor's results would undoubtedly lend greater insight into the effects of weapon-system mobility and attrition-rate dependencies.

V. SUMMARY

Analytic solutions to variable-coefficient Lanchester-type equations were used to examine the effects of attack velocity on the outcome of engagements between two homogeneous forces. Constant and range dependent (linear and quadratic) attrition-rate coefficients were considered. The terminal numerical strength for each force was determined for various values of attack velocity. These terminal strengths were determined for the cases when the opposing forces had the same type of kill-rates, e.g., both linear, and when they had different types of kill-rates.

When the opposing forces had the same type of kill-rates, the Lanchester square law applied. An attacking force with initial numerical superiority, but with firepower inferiority, was able to achieve terminal numerical superiority by attacking at a high speed. This result is a consequence of the Lanchester square law, and the fact that an increase in attack velocity is equivalent to a decrease in the duration of the battle.

When the opposing forces had different types of kill-rates, the effects of attack velocity were dependent on the forms of the kill-rates. If the kill-rate of the attacking force dominated that of the defender, the attacker could conserve his force by attacking at slower speeds. Conversely, if the kill-rate of the defender was dominate, it was advantageous for the attacker to attack at higher speeds.

TABLE I. VALUES OF INPUT PARAMETERS

	Case I	Case II
X Force Initial Strength	30	30
Y Force Initial Strength	20	20
α_0	0.02	0.02
β_0	0.005	0.01
Effective Range, R_e	2000m	2000m
m,n	0,1,2	0,1,2

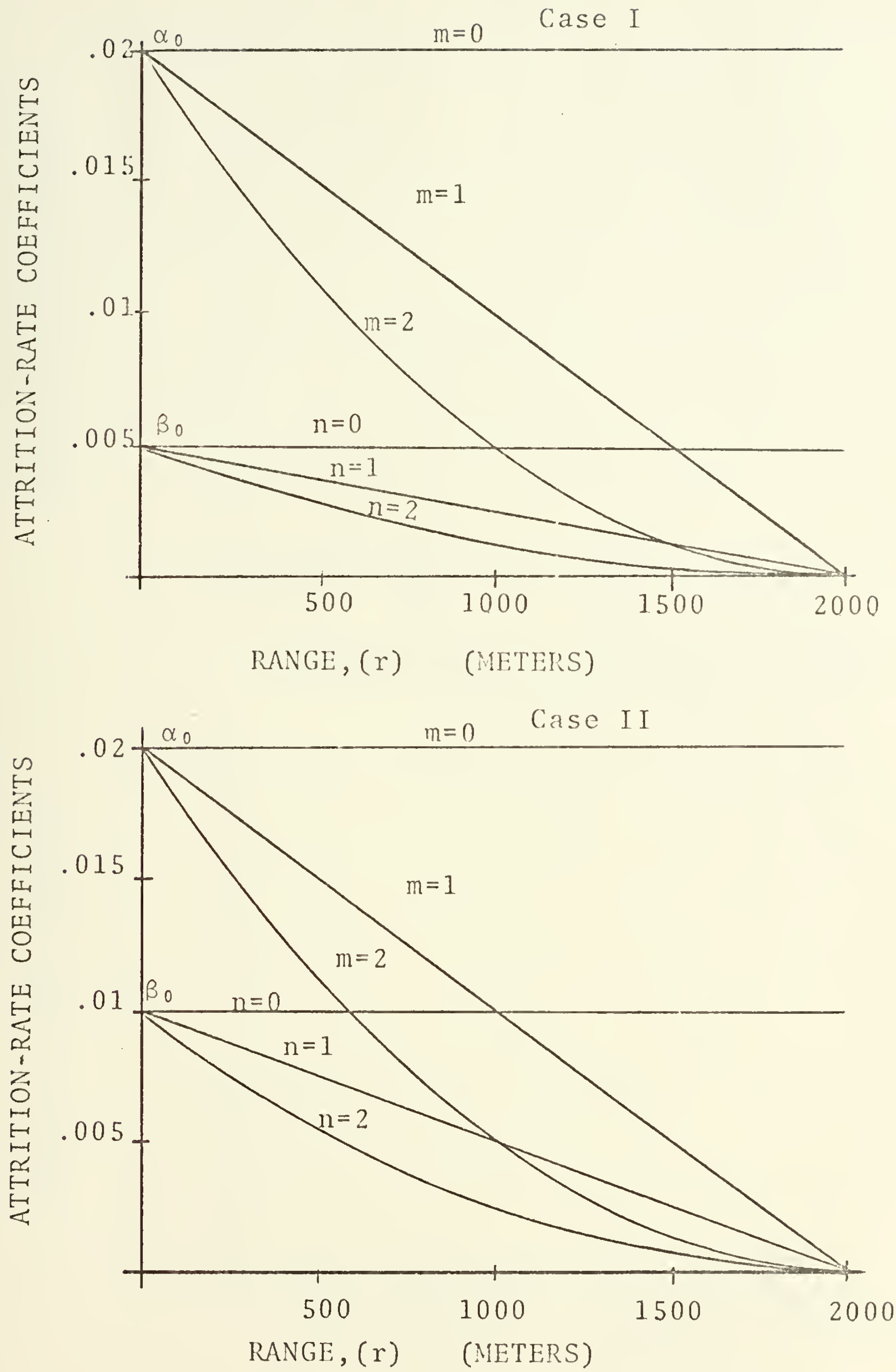


Figure 1. Attrition-rate coefficients, $\alpha(r)$ and $\beta(r)$, as a function of range for various values of m and n.

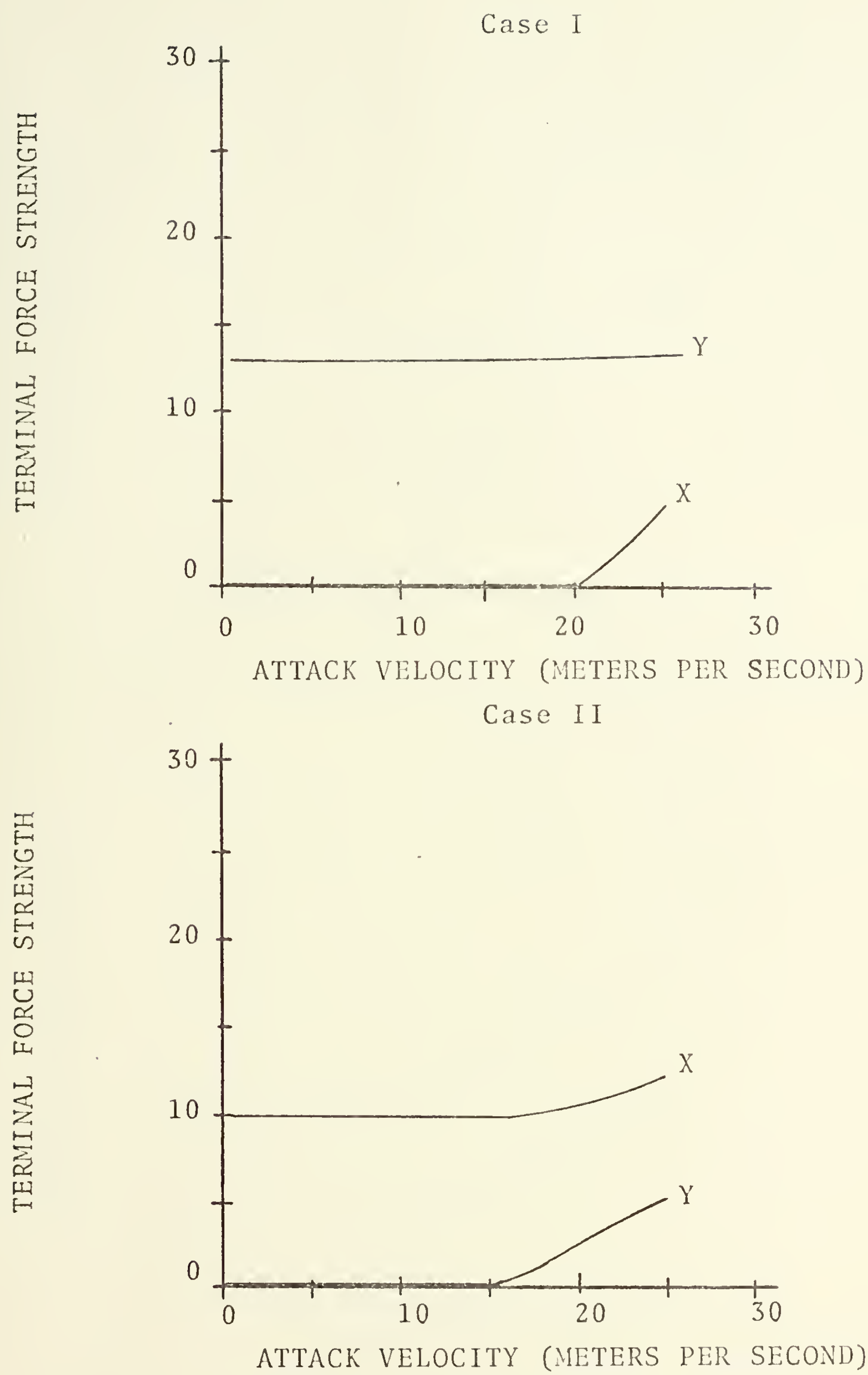


Figure 2. Effect of attack velocity on terminal force strength; m=0, n=0.

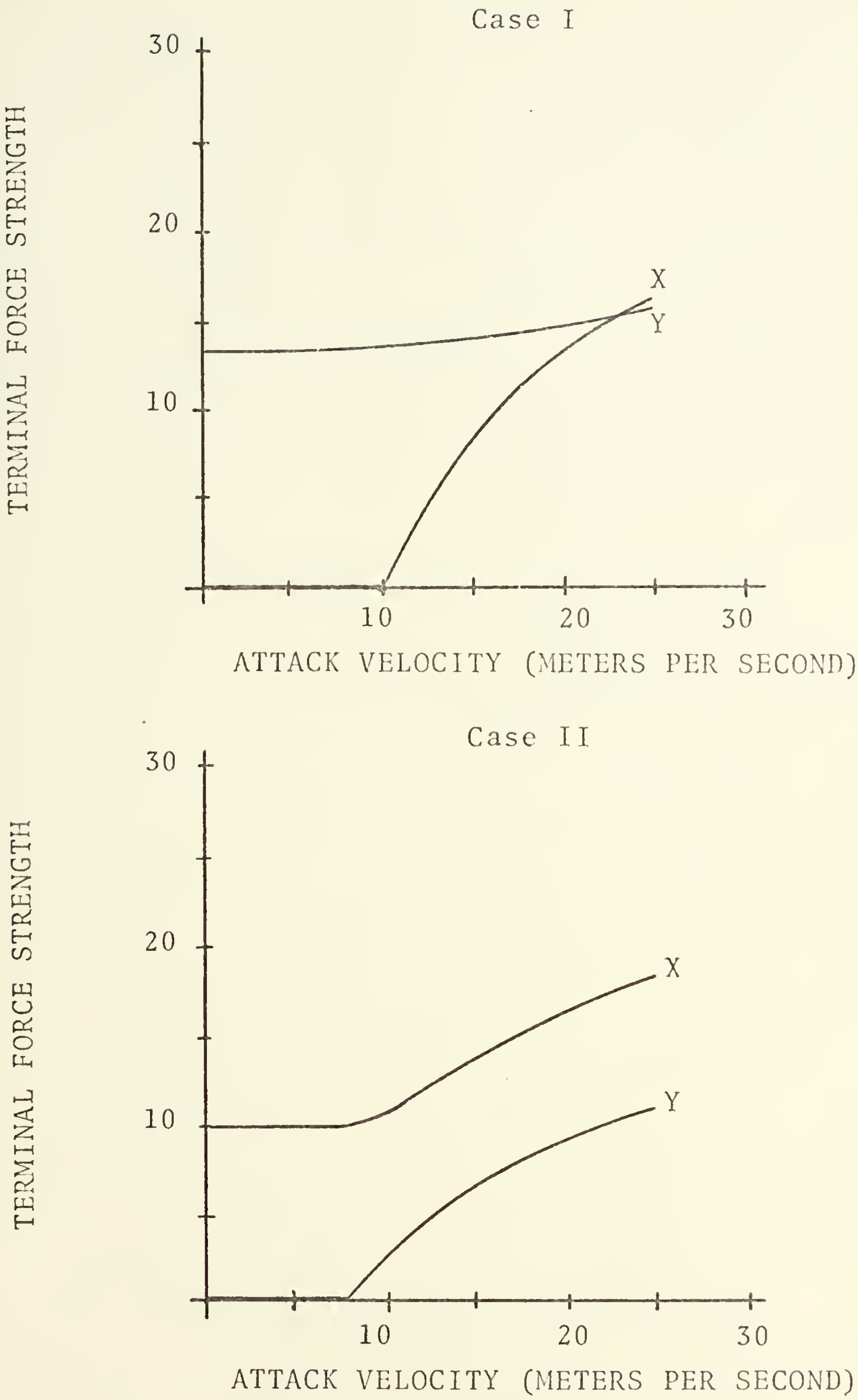


Figure 3. Effect of attack velocity on terminal force strength; m=1, n=1.

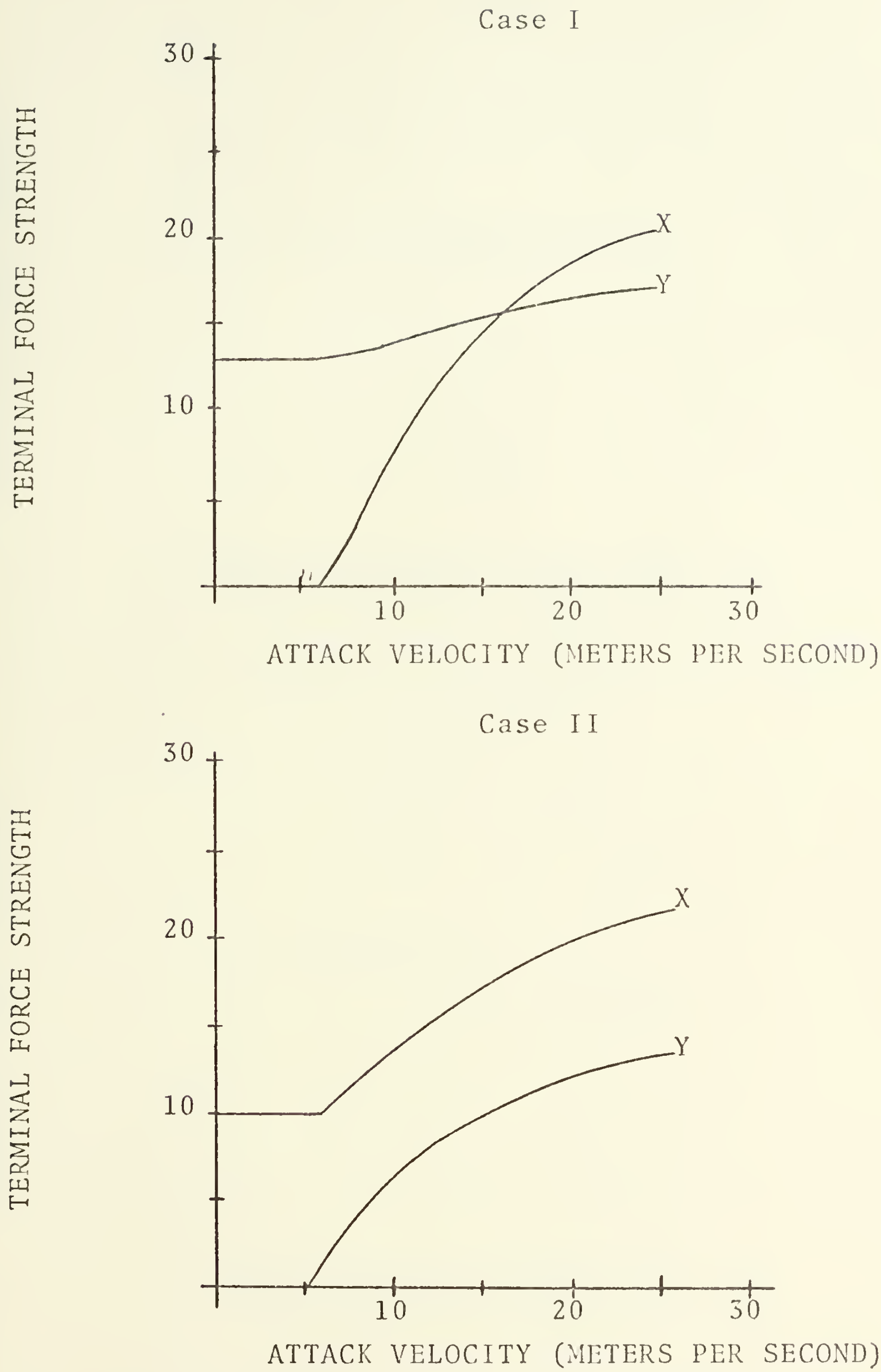
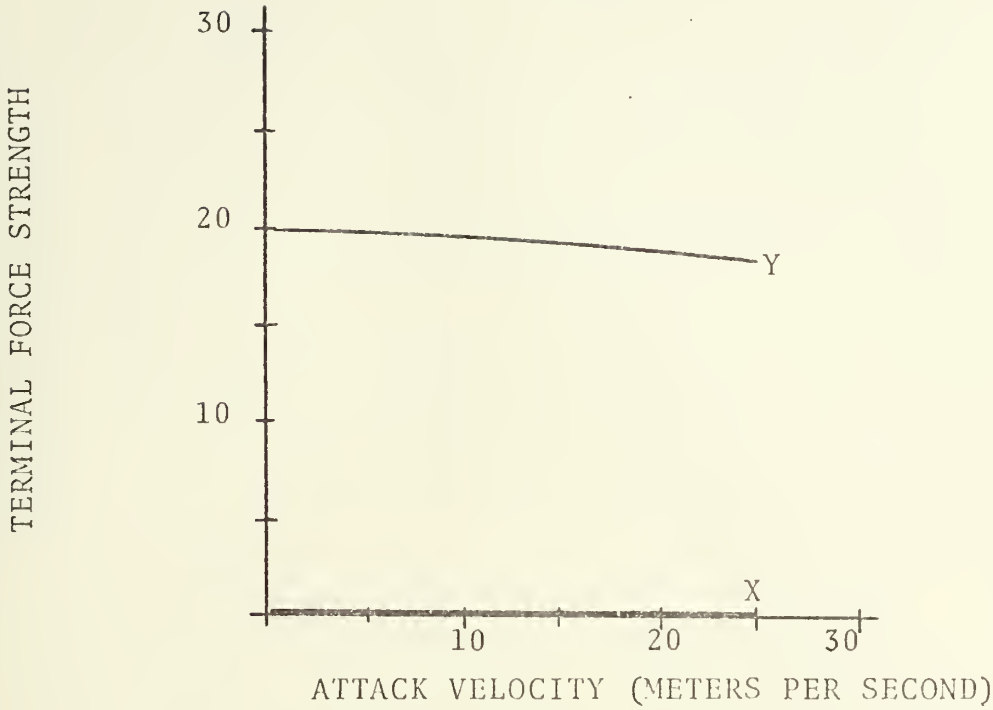


Figure 4. Effect of attack velocity on terminal force strength; m=2, n=2.

Case I



Case II

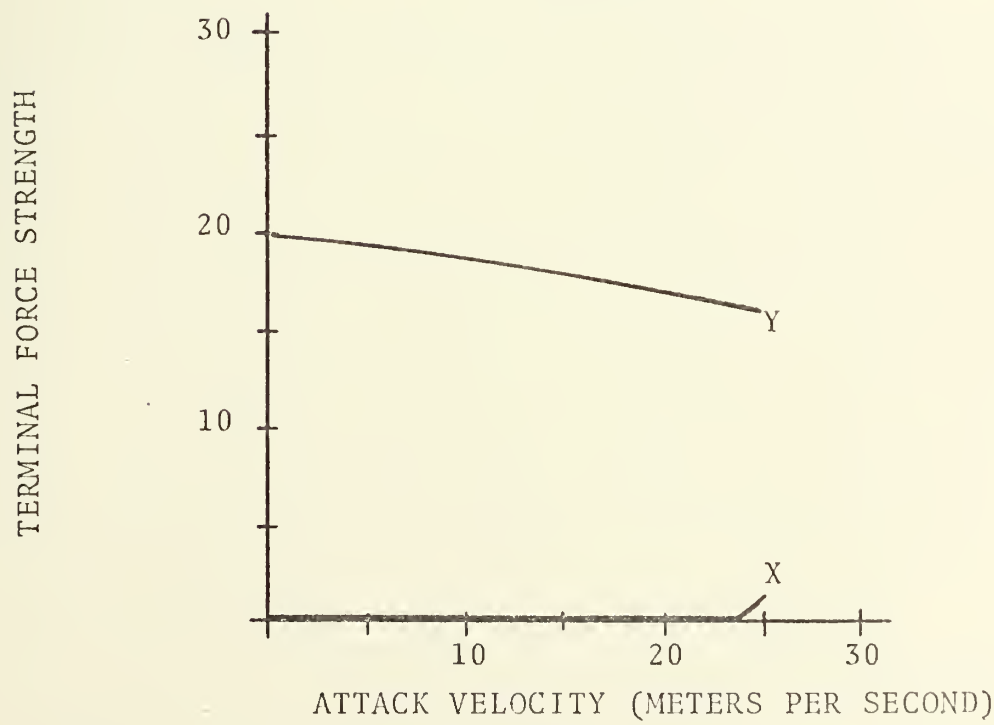


Figure 5. Effect of attack velocity on terminal force strength; m=0, n=1.

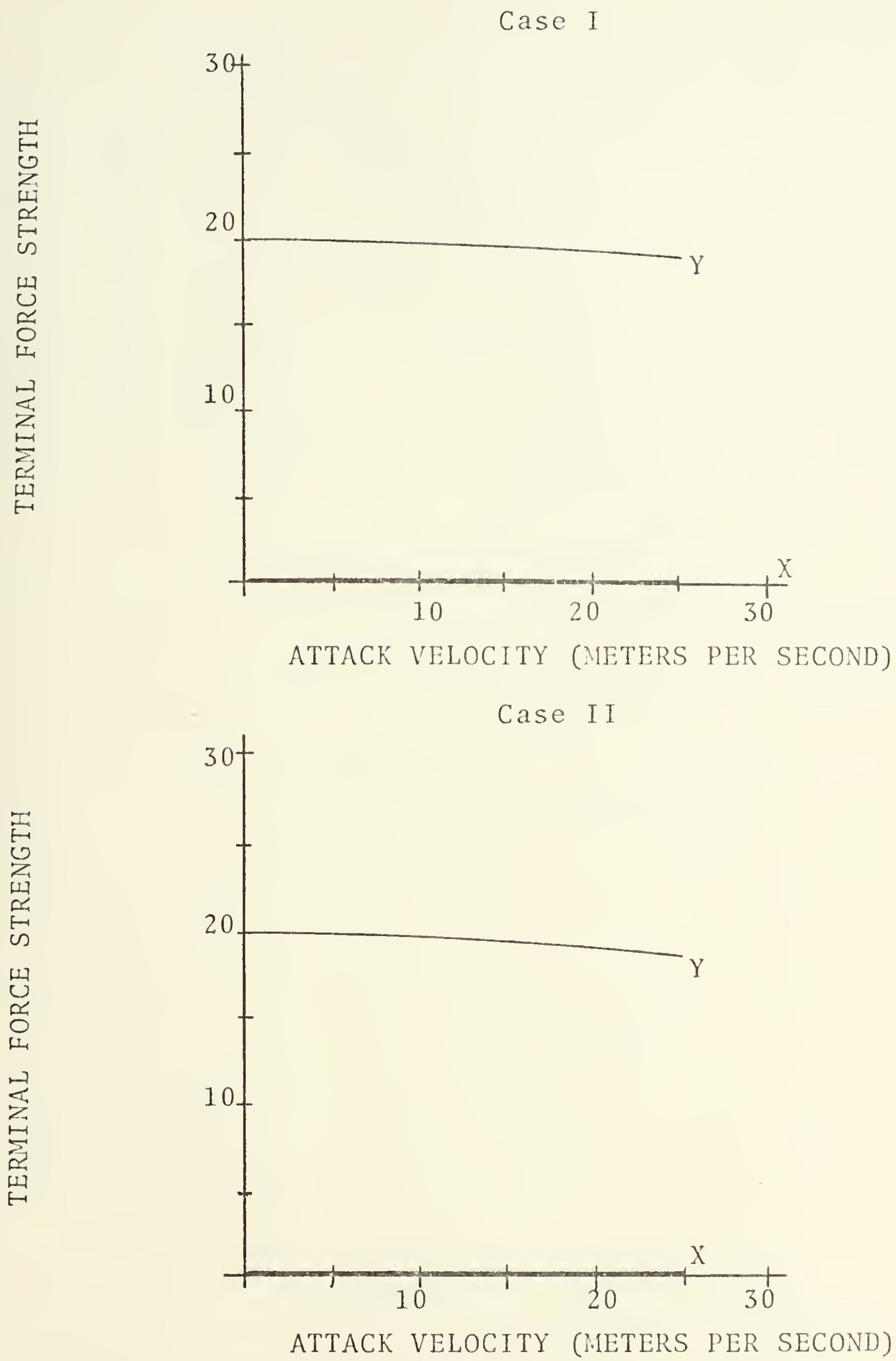


Figure 6. Effect of attack velocity on terminal force strength; m=0, n=2.

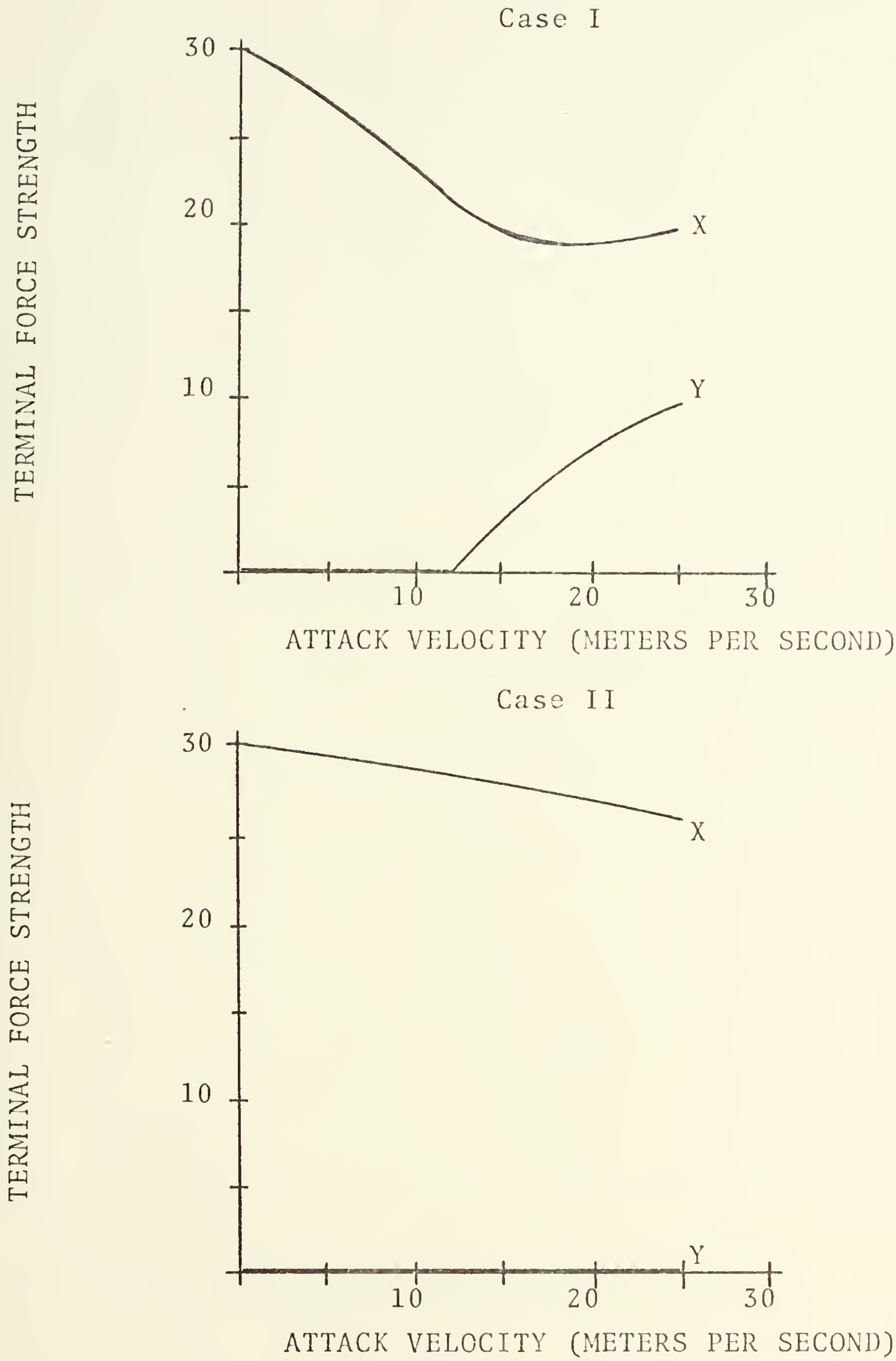


Figure 7. Effect of attack velocity on terminal force strength; m=1, n=0.

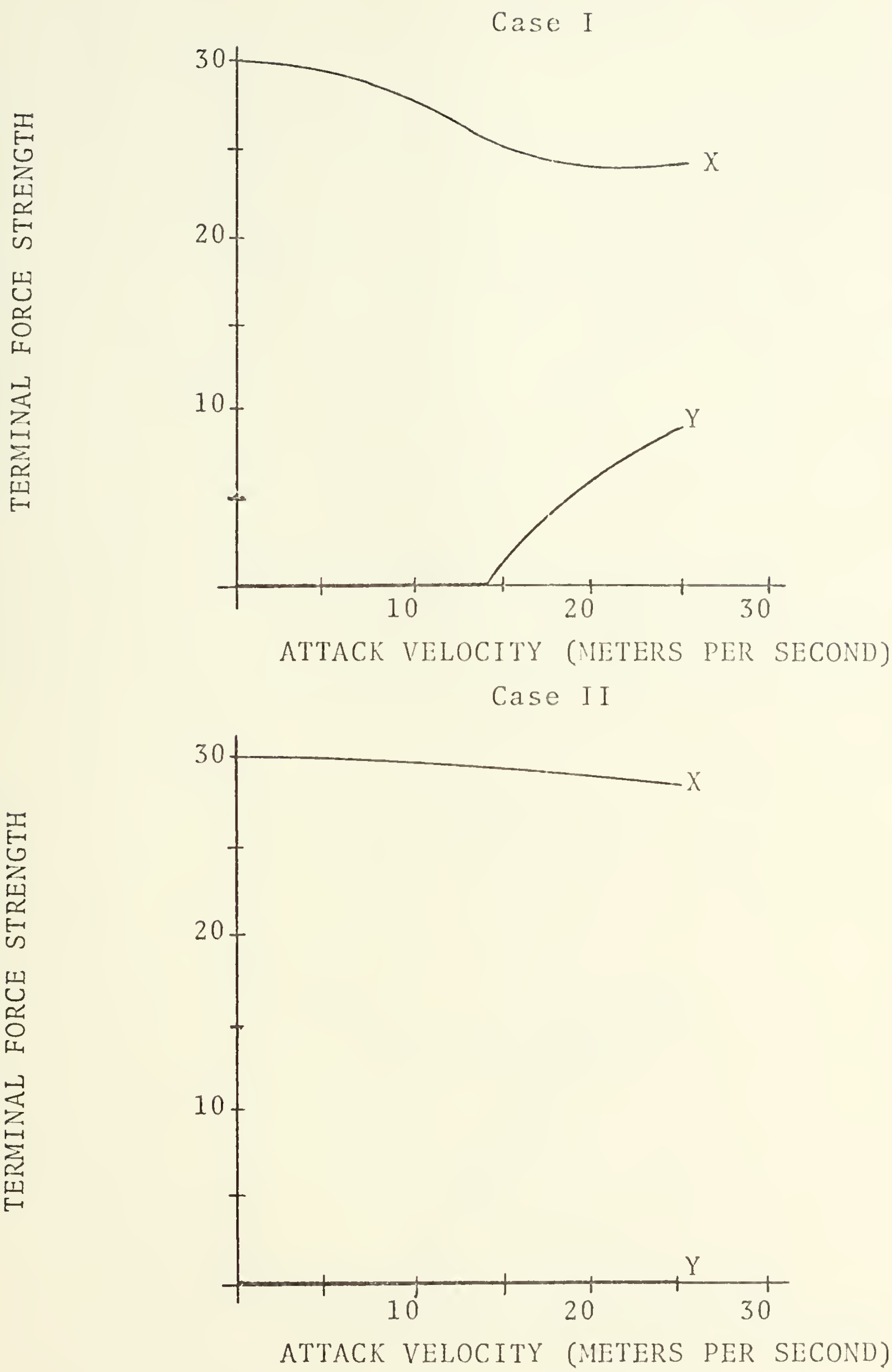


Figure 8. Effect of attack velocity on terminal force strength; m=2, n=0.

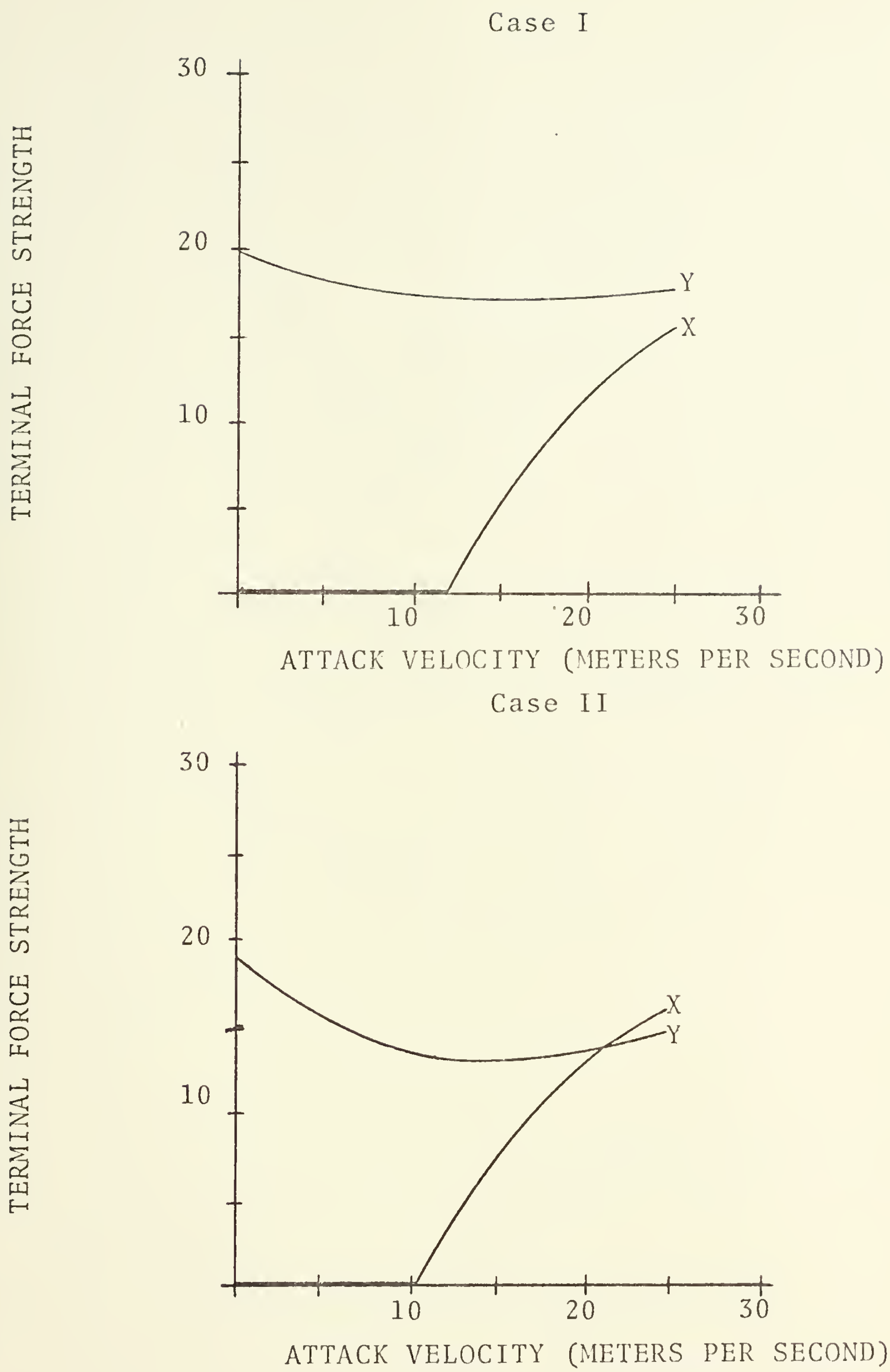


Figure 9. Effect of attack velocity on terminal force strength; m=1, n=2.

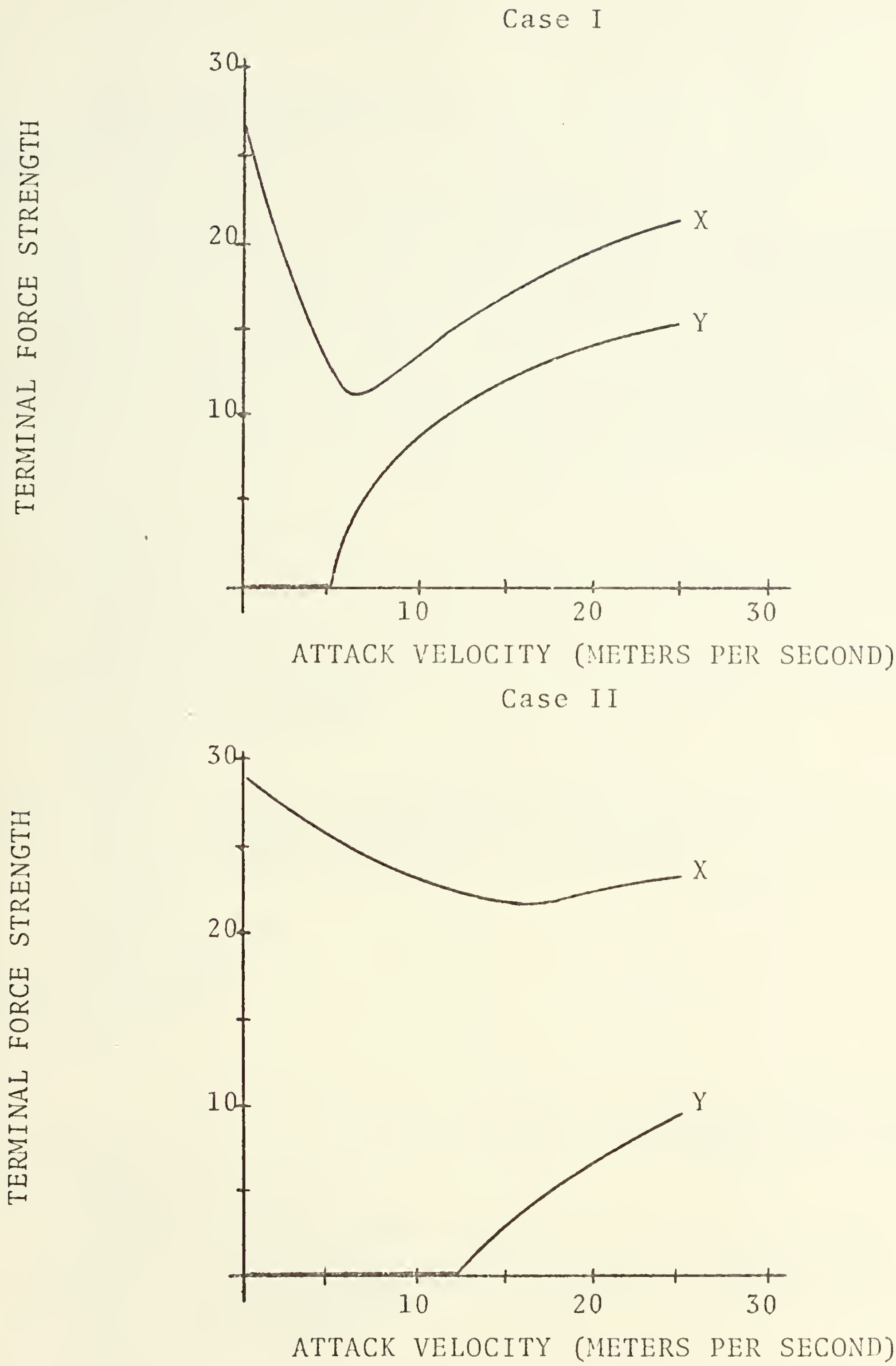


Figure 10. Effect of attack velocity on terminal force strength; m=2, n=1.

PROGRAM TO DETERMINE EFFECT OF ATTACK VELOCITY

```
      CCMMON IPT,IOUT,ISTEP
      DATA H4END/4HEND /
C MAIN PROGRAM TO COMPUTE SERIES. BOTH WEAPON SYSTEMS HAVE THE
C SAME EFFECTIVE RANGE.
   10 FORMAT(2I4)
   20 FORMAT(A4,3X,2F6.1,2F8.4,4I3,2F7.1,F9.1)
   30 FORMAT(1H1,5X,'LANCHESTER ATTRITION THEORY. - BOTH WEA
     1PON SYSTEMS HAVE THE SAME EFFECTIVE RANGE.'///2H ,5X,'
     2CASE NUMBER',I6//1H ,5X,7H     X0=,F12.2/1H ,5X,7H     Y
     30=,F12.2/1H ,5X,7HALPHAO=,F12.4/1H ,5X,7H BETAO=,F12.4
     4/1H ,5X,7H  MMIN=,7X,I5,5X,5HMMAX=,7X,I5/1H ,5X,7H  NM
     5IN=,7X,I5,5X,5HNMAX=,7X,I5/1H ,5X,7HDELTAR=,F12.2/1H ,
     65X,7HRANGEO=,F12.2/1H ,5X,7H      V=,F12.2)
   40 FORMAT(1H1,5X,'LANCHESTER ATTRITION PROFILE'//1H ,9X,'
     1BOTH WEAPON SYSTEMS HAVE THE SAME EFFECTIVE RANGE.'//1
     2H ,8X,'ALPHAO=',F10.4/1H ,9X,'BETAO=',F10.4/1H ,13X,'M
     3=',I5/1H ,13X,'N=',I5/1H ,13X,'V=',F8.2///1H ,5X,'IST
     4EP',4X,'RANGE',7X,'X(R)',6X,'Y(R)',7X,'F(X,Y)',10X,'IS
     5TEP'/)
   50 FCRMAT(1H ,5X,I3,3X,F8.1,F11.4,F10.4,E14.6,6X,I3)
      READ(5,10)IPT,IOUT
      ICASE=0
  300 READ(IPT,20)FLAG,X0,Y0,ALPHA3,BETAO,MMIN,MMAX,NMIN,NMA
     1X,DELTAR,V,RANGEO
      IF(FLAG.EQ.H4END) GO TO 100
      ICASE=ICASE+1
      WRITE(IOUT,30)ICASE,X0,Y0,ALPHAO,BETAO,MMIN,MMAX,NMIN,
     1NMAX,DELTAR,RANGEO,V
      DO 200 M=MMIN,MMAX
      DO 200 N=NMIN,NMAX
      R=RANGEO-DELTAR
      ISTEP=1
      WRITE(IOUT,40) ALPHAO,BETAO,M,N,V
  400 CALL XYSER(X0,Y0,ALPHAO,BETAO,M,N,R,RANGEO,V,X,Y)
      FXY=BETAO*X*X-ALPHAO*Y*Y
      WRITE(IOUT,50) ISTEP,R,X,Y,FXY,ISTEP
      IF(.NOT.((X.GT.0.0).AND.(Y.GT.0.0).AND.(R.GT.0.0))) GO
     1TO 200
      R=R-DELTAR
      ISTFP=ISTEP+1
      GO TO 400
  200 CONTINUE
      GO TO 300
  100 CONTINUE
      END


      SUBROUTINE XYSER(X0,Y0,ALPHAO,BETAO,M,N,R,RANGEO,V,X,Y
     1)
C THIS SUBROUTINE CALCULATES X(R) AND Y(R) ACCORDING TO THEIR
C SERIES SOLUTION.
      IMPLICIT REAL*8(C)
      COMMON IPT,IOUT,ISTEP
   10 FORMAT(1H ,5X,'SERIES FOR X(R) HAS NOT CONVERGED. NEXT
     1ENTRY IS IN ERROR.')
   20 FORMAT(1H ,5X,'SERIES FOR Y(R) HAS NCT CONVERGED. NEXT
     1ENTRY IS IN ERROR.')
      DATA EPS/1.0E-06/
  500 P=(M+1.0)/(M+N+2.0)
      Q=1-P
      S=(M+N+2.)/2.0
      COMP1=((RANGEO*SQRT(ALPHAO*BETAO))/(V*2.*S))**2
      COMP2=1.0-(R/RANGEO)
      CONST1=CCMP1*(COMP2**(2*S))
      CONST2=COMP1*(COMP2**(2*P*S))
      CONST3=CCMP1*(COMP2**(2*Q*S))
C SOLUTICN FOR X(R)
```

```
  600 TERM1=1.0/GAMMA(1.0-P)
      TERM2=(COMP2**(2*P*S))/GAMMA(1.0+P)
      SUM1=TERM1
      SUM2=TERM2
      DO 100 K=1,100
      TERM1=TERM1*CONST1*GAMMA(K-P)/(K*GAMMA(K+1.0-P))
      TERM2=TERM2*CONST2*GAMMA(K+P)*(COMP2**(2*K*S))/(K*GAMM
     1A(K+1.0+P)*COMP2**(2*(K-1)*S+2*P*S))
      SUM1=SUM1+TERM1
      SUM2=SUM2+TERM2
      IF(AMAX1(ABS(TERM1),ABS(TERM2)).LT.EPS) GO TO 200
  100 CONTINUE
      WRITE(IOUT,10)
  200 X=X0*GAMMA(1.0-P)*SUM1-RANGE0*ALPHA0*Y0*GAMMA(1.0+P)*S
     1UM2/(V*(M+1.0))
SOLUTION FOR Y(R)
      TERM3=1.0/GAMMA(1.0-Q)
      TERM4=COMP2**(2*Q*S)/GAMMA(1.0+Q)
      SUM3=TERM3
      SUM4=TERM4
      DO 300 J=1,100
      TERM3=TERM3*CONST1*GAMMA(J-Q)/(J*GAMMA(J+1.0-Q))
      TERM4=TERM4*CONST3*GAMMA(J+Q)*COMP2**(2*J*S)/(J*GAMMA(
     1J+1.0+Q)*COMP2**(2*(J-1)*S+2*Q*S))
      SUM3=SUM3+TERM3
      SUM4=SUM4+TERM4
      IF(AMAX1(ABS(TERM3),ABS(TERM4)).LT.EPS) GO TO 400
  300 CONTINUE
      WRITE(IOUT,20)
  400 Y=Y0*GAMMA(1.0-Q)*SUM3-RANGE0*BETA0*X0*GAMMA(1.0+Q)*SU
     1M4/(V*(N+1.0))
      RETURN
      END
```

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13. ABSTRACT

This thesis examines the effect of attack velocity on the outcome of Lanchester-type engagements between forces with range dependent kill-rates. Range dependent (linear and quadratic) kill-rates are considered, and analytic solutions to Lanchester-type equations are utilized in this study.

By varying the attack velocity, the effects on terminal force strengths are investigated for the case when an attacking force has the initial fighting strength superiority, and for the case when a defending force has the initial fighting strength superiority.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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Variable attrition-rate coefficient						
Attack velocity						
Range dependent kill-rate						

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